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LETTER TO THE EDITOR

The gauge equivalence of the Davey–Stewartson equation and (2+1)-dimensional continuous Heisenberg ferromagnetic model

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Abstract. The gauge equivalence between the Davey-Stewartson (DS) equation and the (2+1)-dimensional continuous Heisenberg ferromagnetic model is shown explicitly. Through the gauge transformation, solutions of the DSII are obtained from the vortex type solutions of the former equation.

In the study of (1+1)-dimensional nonlinear evolution equations, a well known gauge equivalence takes place between the continuous Heisenberg ferromagnet equation and the nonlinear Schrödinger equation [1].

The analogue of the first equation in (2+1) dimensions has been given by Ishimori [2] and it takes the form

$$\boldsymbol{S}_t + \boldsymbol{S} \times (\boldsymbol{S}_{xx} + \alpha^2 \boldsymbol{S}_{yy}) + \phi_x \boldsymbol{S}_y + \phi_y \boldsymbol{S}_x = 0 \tag{1a}$$

$$\phi_{xx} - \alpha^2 \phi_{yy} + 2\alpha^2 \boldsymbol{S}(\boldsymbol{S}_x \times \boldsymbol{S}_{yy}) = 0 \tag{1b}$$

with $\alpha^2 = -1$, $\boldsymbol{S} = \boldsymbol{S}(x, y, t) = (s_1(x, y, t), s_2(x, y, t), s_3(x, y, t))$ satisfying $\boldsymbol{S} \cdot \boldsymbol{S} = 1$, and $S_t = \partial S/\partial t$, etc. A (2+1)-dimensional analogue of the nonlinear Schrödinger equation is called the Davey-Stewartson (DS) equation, which can be obtained from the following system

$$\begin{aligned} &\mathrm{i}q_t - q_{xx} - \alpha^2 q_{yy} + 2q\psi = 0 \\ &\mathrm{i}r_t + r_{xx} + \alpha^2 r_{yy} - 2r\psi = 0 \end{aligned} \tag{2a}$$

$$\psi_{yy} - \alpha^2 \psi_{xx} + (qr)_{xx} + \alpha^2 (qr)_{yy} = 0 \tag{2b}$$

by letting $r = \bar{q}$, where $q = q(x, y, t)$, etc, and the 'bar' denotes the complex conjugate.

The question then arises whether the gauge equivalence can also take place between (1) and the DS equation. In [3], the positive answer to the above question has been mentioned but not explicitly given. The purpose of this letter is to show such equivalence in detail. More generally, we first construct the gauge transformation from (1) to (2). By letting $\alpha = \mathrm{i}$, it becomes the transformation from (1) with $\alpha = \mathrm{i}$ to the DSII equation (i.e. (2) with $\alpha = \mathrm{i}$), therefore the vortex type solutions of (1) with $\alpha = \mathrm{i}$, found in [2], can be transformed to solutions of DSII. Finally, we will show that the reversible gauge transformation from DSII to (1) with $\alpha = \mathrm{i}$ is also valid.

Let us first write down the Lax pairs of both (1) and (2); they are (see [2-4]):

$$L_1 = \alpha \partial_y + S\partial_x \tag{3a}$$

$$L_2 = \partial_t - 2\mathrm{i}S\partial_x^2 - (\mathrm{i}S_x + \mathrm{i}\alpha S_y S + \alpha^3 \phi_x S - \phi_y)\partial_x \tag{3b}$$

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for (1) with

$$S=\begin{pmatrix} s_3 & \bar{s} \\ s & -s_3 \end{pmatrix} \qquad s=s_1+\mathrm{i}s_2,\ S^2=I \tag{4}$$

and

$$\bar{L}_1=\alpha\partial_y-\sigma_3\partial_x+Q \tag{5a}$$

$$\bar{L}_2=\partial_t+2\mathrm{i}\sigma_3\partial_x^2-2\mathrm{i}Q\partial_x+C \tag{5b}$$

for (2), where σ_3 is the Pauli matrix and

$$Q=\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \qquad C=\begin{pmatrix} 0 & q_x+\alpha q_y \\ r_x-\alpha r_y & 0 \end{pmatrix}+\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{6}$$

$$(a-b)_x-\alpha(a+b)_y=2\mathrm{i}(qr)_x \tag{7a}$$

$$(a+b)_x-\alpha(a-b)_y=2\mathrm{i}\alpha(qr)_y. \tag{7b}$$

Therefore, (1) and (2) can be represented as $[L_1, L_2]=0$, and $[\bar{L}_1, \bar{L}_2]=0$, respectively. If there exists a gauge transformation such that

$$\bar{L}_1=TL_1T^{-1} \qquad \bar{L}_2=TL_2T^{-1} \tag{8}$$

then one finds that $[\bar{L}_1, \bar{L}_2]=T[L_1, L_2]T^{-1}$, which indicates the equivalence of (1) and (2). By comparing coefficients of ∂_x^i on both sides of (8), such a matrix T must satisfy

$$\sigma_3=-TST^{-1} \tag{9}$$

$$Q=-\alpha T_yT^{-1}-TST^{-1}T_xT^{-1} \tag{10}$$

$$2\mathrm{i}\sigma_3=-2\mathrm{i}TST^{-1} \tag{11}$$

$$2\mathrm{i}Q=4\mathrm{i}\sigma_3T_xT^{-1}+T(\mathrm{i}S_x-\mathrm{i}\alpha SS_y+\alpha^3\phi_xS-\phi_y)T^{-1} \tag{12}$$

$$C=-T_tT^{-1}-2\mathrm{i}\sigma_3T_{xx}T^{-1}+2\mathrm{i}QT_xT^{-1}. \tag{13}$$

Let us now consider the transformation from (1) to (2). By solving (9) and (11), the general form of T is

$$T=\mathrm{diag}(\lambda, \mu)(S-\sigma_3) \tag{14}$$

where λ and μ are temporally arbitrary. Substitute (14) into (10); require that the right-hand side of (10) be off-diagonal; then constraints for λ and μ appear as

$$\left(\frac{\lambda_x}{\lambda}+\frac{\bar{s}_xs}{2(1-s_3)}-\frac{s_{3x}}{2}\right)-\alpha\left(\frac{\lambda_y}{\lambda}+\frac{\bar{s}_ys}{2(1-s_3)}-\frac{s_{3y}}{2}\right)=0 \tag{15}$$

$$\left(\frac{\mu_x}{\mu}+\frac{s_x\bar{s}}{2(1-s_3)}-\frac{s_{3x}}{2}\right)+\alpha\left(\frac{\mu_y}{\mu}+\frac{s_y\bar{s}}{2(1-s_3)}-\frac{s_{3y}}{2}\right)=0 \tag{16}$$

and Q is then exactly given in terms of S and λ, μ in the following way:

$$q=\frac{\lambda}{\mu}\left[\left(\frac{\bar{s}s_{3x}}{2(1-s_3)}+\frac{\bar{s}_x}{2}\right)-\alpha\left(\frac{\bar{s}s_{3y}}{2(1-s_3)}+\frac{\bar{s}_y}{2}\right)\right] \tag{17a}$$

$$r=\frac{\mu}{\lambda}\left[\left(\frac{ss_{3x}}{2(1-s_3)}+\frac{s_x}{2}\right)+\alpha\left(\frac{ss_{3y}}{2(1-s_3)}+\frac{s_y}{2}\right)\right] \tag{17b}$$

$$qr=\tfrac{1}{4}[\boldsymbol{S}_x^2-\alpha^2\boldsymbol{S}_y^2-2\mathrm{i}\alpha\boldsymbol{S}(\boldsymbol{S}_x\times\boldsymbol{S}_y)]. \tag{18}$$

Substituting T into (14) and Q, given by (17), into (12) gives rise to additional constraints on λ and μ and, for instance, we have

$$2\mathrm{i}\left(\frac{\lambda_x}{\lambda}+\frac{\bar{s}_x s}{2(1-s_3)}-\frac{S_{3x}}{2}\right)+2\mathrm{i}\alpha\left(\frac{\lambda_y}{\lambda}+\frac{\bar{s}_y s}{2(1-s_3)}-\frac{s_{3y}}{2}\right)-\alpha^3\phi_x-\phi_y=0 \qquad (19)$$

and a similar equation for μ.

Equations (15) and (19) are equivalent to

$$4\mathrm{i}\left(\frac{\lambda_x}{\lambda}+\frac{\bar{s}_x s}{2(1-s_3)}-\frac{s_{3x}}{2}\right)-\alpha^3\phi_x-\phi_y=0 \qquad (20a)$$

$$4\mathrm{i}\alpha\left(\frac{\lambda_y}{\lambda}+\frac{\bar{s}_y s}{2(1-s_3)}-\frac{s_{3y}}{2}\right)-\alpha^3\phi_x-\phi_y=0 \qquad (20b)$$

and similarly, we have the equivalent constraints for μ:

$$-4\mathrm{i}\left(\frac{\mu_x}{\mu}+\frac{s_x\bar{s}}{2(1-s_3)}-\frac{s_{3x}}{2}\right)+\alpha^3\phi_x-\phi_y=0 \qquad (21a)$$

$$4\mathrm{i}\alpha\left(\frac{\mu_y}{\mu}+\frac{s_y\bar{s}}{2(1-s_3)}-\frac{s_{3y}}{2}\right)+\alpha^3\phi_x-\phi_y=0. \qquad (21b)$$

By direct, but lengthy, calculation one finds that equation (1b) plays the role of the compatibility conditions for both (20) and (21).

For (13), we only need to check that the right-hand side matrix, still defined as C, has the form of (6) and (7) with Q given by (17). Notice that

$$T_tT^{-1}=\mathrm{diag}\left(\frac{\lambda_t}{\lambda},\frac{\mu_t}{\mu}\right)+\mathrm{diag}(\lambda,\mu)S_t(S-\sigma_3)^{-1}\,\mathrm{diag}(\lambda^{-1},\mu^{-1}). \qquad (22)$$

Hence, substituting (1) into (22), one can check that the off-diagonal part of C is $-\mathrm{i}(Q_x+\alpha\sigma_3Q_y)$ with Q given by (17) and the diagonal entries of C, say a and b, apparently depend on λ_t and μ_t. By using

$$\frac{\lambda_x}{\lambda}-\frac{\mu_x}{\mu}=\frac{1}{2\mathrm{i}}\phi_y-\mathrm{i}\left(\frac{s_{1x}s_2-s_{2x}s_1}{1-s_3}\right) \qquad (22a)$$

$$\alpha\left(\frac{\lambda_y}{\lambda}+\frac{\mu_y}{\mu}\right)=\frac{1}{2\mathrm{i}}\phi_y+\alpha\left(\frac{s_{3y}}{1-s_3}\right) \qquad (22b)$$

obtained directly from (20) and (21), however, we find that

$$(a-b)_x-\alpha(a+b)_y=\tfrac{1}{2}\mathrm{i}(\boldsymbol{S}_x^2-\alpha^2\boldsymbol{S}_y^2-2\mathrm{i}\alpha(\boldsymbol{S}(\boldsymbol{S}_x\times\boldsymbol{S}_y))_x=2\mathrm{i}(qr)_x \qquad (23)$$

i.e., a and b satisfy (7a), and similarly, we can verify that a and b also satisfy (7b).

Therefore, we have given the gauge transformation from (1) to (2) explicitly. In particular, when $\alpha^2=-1$, from (20) and (21) we have $\mu=\bar{\lambda}$ and then $r=\bar{q}$ in (17); thus the gauge transformation also takes place from (1) (with $\alpha^2=-1$) to the equation of DSII.

It is known that the vortex type solutions of (1) with $\alpha=\mathrm{i}$ are [2]

$$s=\frac{2\bar{f}g}{\Delta} \qquad s_3=\frac{f\bar{f}-g\bar{g}}{\Delta} \qquad \Delta=f\bar{f}+g\bar{g} \qquad (24)$$

where $f=f(z,t)$, $g=g(z,t)$, $z=x+\mathrm{i}y$ and they satisfy

$$f_t+\mathrm{i}f_{zz}=g_t+\mathrm{i}g_{zz}=0. \qquad (25)$$

Taking $\alpha=\mathrm{i}$ and substituting (24) into (20) and (21), λ and μ can be solved as

$$\lambda=\bar{g}^{-1} \qquad \mu=\bar{\lambda}=g^{-1} \tag{26}$$

thus the solutions of DSII read

$$q=\frac{2}{\Delta}(f_z g-g_z f) \qquad r=\bar{q} \tag{27}$$

with f and g satisfying (25). For example, taking the solution of (25) in the form

$$g=1 \qquad f=(z^2-2\mathrm{i}t) \tag{28}$$

then the solution of DSII reads

$$q=\frac{2z}{(z\bar{z})^2+2\mathrm{i}(z^2-\bar{z}^2)t-4t^2+1} \tag{29}$$

with $z=x+\mathrm{i}y$, $\bar{z}=x-\mathrm{i}y$.

To construct the gauge transformation from DSII to (1) with $\alpha=\mathrm{i}$, let us choose the unitary matrix solution of

$$\bar{L}_1 T=0 \qquad \bar{L}_2 T=0. \tag{30}$$

Then define S as

$$S=-T^{-1}\sigma_3 T \tag{31}$$

i.e. T satisfies (9)–(11) and (13) with Q having been given. To determine ϕ, such that (12) and (1b) are valid for the given Q, and S in (31), we substitute (31) into (12) and using (10) and (13), we find

$$\mathrm{i}\{\sigma_3, T_x T^{-1}\}+\mathrm{i}\alpha\sigma_3\{\sigma_3, T_y T^{-1}\}-\alpha^3\phi_x\sigma_3-\phi_y=0 \tag{32}$$

where { , } denotes the anticommutator of two matrices. Therefore ϕ_x, ϕ_y are given by

$$\phi_x=\tfrac{1}{2}\mathrm{i}\,\mathrm{tr}(\alpha\sigma_3\{\sigma_3, T_x T^{-1}\}+\alpha^2\{\sigma_3, T_y T^{-1}\}) \tag{33a}$$

$$\phi_y=\tfrac{1}{2}\mathrm{i}\,\mathrm{tr}(\{\sigma_3, T_x T^{-1}\}+\alpha\sigma_3\{\sigma_3, T_y T^{-1}\}). \tag{33b}$$

On the other hand, using (10), we have

$$Q_x+\alpha\sigma_3 Q_y=\sigma_3((T_x T^{-1})_x-\alpha^2(T_y T^{-1})_y)+\alpha[T_y T^{-1}, T_x T^{-1}]. \tag{34}$$

It immediately follows that

$$\mathrm{tr}\,\sigma_3\{\sigma_3, \alpha((T_x T_x^{-1}-\alpha^2(T_y T^{-1})_y)\}=0. \tag{35}$$

One can easily check that the above equation is the compatibility condition of (33). Hence from (33), we have

$$\phi_{xx}-\alpha^2\phi_{yy}=\tfrac{1}{2}\mathrm{i}\alpha^2\,\mathrm{tr}\{\sigma_3, [T_x T^{-1}, T_y T^{-1}]\}=0. \tag{36}$$

This equation can be written as

$$\phi_{xx}-\alpha^2\phi_{yy}=\mathrm{i}\alpha^2\,\mathrm{tr}\,S[S_x, S_y] \tag{37}$$

with S given by (31). Equation (37) is the same as (1b) but in a different form. Therefore, we have proved that the matrix T satisfying (30) gives rise to the gauge transformation from DSII to (1) with $\alpha=\mathrm{i}$.

In summary, we constructed the gauge transformation from (1) to (2), which includes the transformation from (1) with $\alpha = \mathrm{i}$ to DSII, hence solutions of DSII can be obtained from the vortex type solutions of (1) with $\alpha = \mathrm{i}$ through the transformation. The reversible gauge transformation from DSII to (1) with $\alpha = \mathrm{i}$ is also valid. In this sense, the DSII equation is the gauge equivalent to (1) with $\alpha = \mathrm{i}$.

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References

[1] Zakharov V E and Takhtajan L A 1979 *Theor. Math. Phys.* **38** 26
[2] Ishimori Y 1984 *Prog. Theor. Phys.* **72** 33
[3] Konopelchenko B G and Matkarimov B T 1988 On the inverse scattering transform for the Ishimori equation *Preprint* Universita Degli Studi di Lecce
[4] Fokas A S and Ablowitz M J 1984 *J. Math. Phys.* **25** 2505